Maxwell's Equations in Two Dimension

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Abstract:

Maxwell's equations are the fundamental equations of the electromagnetic field. Before Maxwell's work electrodynamics was comprised of the following experimental observations:

1) Coulomb's law - the elecrostatic inverse square law of force between charges
2) Faraday's law - that a changing magnetic field causes electric effects
3) the absence of magnetic monopoles - ie unlike the electric charge, the source of the magnetic field is always found to be a dipolar
4) Ampere's law - the connection between a current and the magnetic field it causes.

James Clerk Maxwell wrote down all these laws as elegant mathematic equations between the electric field E, and the magnetic field B, and purely for reasons of mathematical symmetry added a hitherto unobserved term to his fourth equation, describing the creation of a magnetic field by a changing electric one. This expanded equation set allowed for a new type of solution - since a changing B field can create a changing E field (Faraday's law) which in turn can create a changing B field (Maxwell's new term) reinforcing the original field, there is a wave solution, wherein energy is carried great distances from the original sources, even in a vacuum. Upon calculating the velocity of propogation of these waves, Maxwell was surprised to find it to be close to the then accepted value for the speed of light, and concluded that light is a type of electromagnetic radiation, thus uniting optics and electrodynamics.

A priori, there are many nonequivalent ways to make the transition from "sterlversal" to "planiversal" electrodynamics. One could search for analogs of:

1) the significant phenomena - ie one would "imagine" how the experimental observations would look, Coulomb's inverse square law would no doubt be replaced by an inverse linear law; electromagnetic waves of some type would be postulated, and so on;
2) the integral (large scale) formulation of the field equations (see Two Dimensional Science and Technology pp32-34;)
3) the differential (local) formulation - we take this approach here;
4) one of the relativistically covariant equation sets - such as the equations for the four by four antisymmetric F-matrix of fields, or for the four potential A;
5) a least action principle - that is, a law stating that the fields will arrange themselves such that some function is minimized; etc. etc.

In this article we will put forth a consistent formulation of electromagnetism in the planiverse, a formulation which parallels the familiar differential treatment of sterlversal electromagnetism, and which leads to the two dimensional analog of the most significant result of Maxwell's work - the prediction of electromagnetic waves.

For comparison purposes we reproduce here the 3d and 2d equation sets:

3d

$\nabla \cdot \underline{E} = 4\pi\rho$	Coulomb - Gauss Law
$\nabla \times \underline{E} = -\frac{1}{c}\frac{\delta B}{\delta t}$	Faraday - Lenz Law
$\nabla \cdot \underline{B} = 0$	Magnetic monopole Law
$\nabla \times \underline{B} = \frac{4\pi}{c}\underline{j} + \frac{1}{c}\frac{\delta E}{\delta t}$	Ampère - Maxwell Law

2d

$II \odot \underline{E} = 2\pi\sigma$	Inverse - Linear - Gauss Law
$II \otimes \underline{E} = -\frac{1\delta B}{c\delta t}$	Scalar - Magnetic - Faraday - Lenz Law
$II \longrightarrow B = \frac{2\pi}{c}\underline{j} + \frac{1}{c}\frac{\delta E}{\delta t}$	Ampère - Stein Law

I - 2d Vector Analysis

Let us begin by reviewing (and creating whenever necessary) some vector analysis in the plane. A 2-vector $\underline{v}$ is an ordered pair of numbers (v_x, v_y) that undergoes the transformation $\underline{v}' = R_\theta \underline{v}$ where $R\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ upon rotation by angle θ, (the inversion $\underline{r} \longrightarrow -\underline{r}$ is proper and exactly equivalent to R_π, while mirror reflections are improper).

We can define two scalar products: (the symmetry properties refer to interchanging $\underline{v}$ and $\underline{w}$)

$\underline{v} \odot \underline{w} = v_x w_x + v_y w_y$ — the symmetric scalar product

$\underline{v} \otimes \underline{w} = v_x w_y - v_y w_x$ — the antisymmetric scalar product.

It is left to the reader to demonstrate that these are true scalars, while the other two symmetric linear combinations are not.

While vector products can be defined, they are not natural since two vectors do not define a truly unique direction in the plane. However, as is well known, the three dimensional "vector product" isn't really a vector at all (rather a "pseudovector", ie like a vector for proper transformations but not for inversions) it is in actuality an antisymmetric second order tensor. Such objects have $\frac{1}{2}n(n-1)$ elements, and thus only for n=3 can we define a "vector product"; for n=2 the analog has one element and is indeed our antisymmetric scalar product.

We define a 2-vector differential operator $II = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}\right)$ in Cartesian coordinates, and analogously to del we produce

$IIs = \left(\frac{\delta s}{\delta x}, \frac{\delta s}{\delta y}\right)$ $\qquad II \odot \underline{v} = \frac{\delta v_x}{\delta x} + \frac{\delta v_y}{\delta y}$

$II \otimes \underline{v} = \frac{\delta v_y}{\delta x} - \frac{\delta v_x}{\delta y}$ $\qquad II^2 s = II \odot IIs = \frac{\delta^2 s}{\delta x^2} + \frac{\delta^2 s}{\delta y^2}$

In addition we define the 2-vector $II \longrightarrow s = R_{\frac{1}{2}\pi} IIs = \left(\frac{\delta s}{\delta y}, -\frac{\delta s}{\delta x}\right)$.

Now, first let's do II x to the last equation, and use the identity

$II \otimes II \rightarrow s = -II^2 s$. We find:

$$-II^2 B = II \otimes II \rightarrow B = II \otimes \frac{1}{c}\frac{\delta E}{\delta t} = \frac{1}{c}\frac{\delta}{\delta t}(II \times \underline{E}) = \frac{1}{c}\frac{\delta}{\delta t}\left(-\frac{1}{c}\frac{\delta B}{\delta t}\right)$$

i.e.

$$II^2 B - \frac{1}{c^2}\frac{\delta^2}{\delta t^2} B = 0$$

which is the classical wave equation for a scalar.

Next take II of the first equation and subtract $II \rightarrow$ of the second. Leaving out the purely algebraic steps, we get

$$II^2 \underline{E} - \frac{1}{c^2}\frac{\delta^2}{\delta t^2}\underline{E} = 0$$

which is the classical wave equation for a 2-vector.

Thus we see that $\underline{E}$ and B as predicted, propagate as two dimensional waves - perhaps detectable to planiversal creatures as 2d light, 2d radio, 2d X-rays, etc.