Finding poles and zeros of a filter.

Let's start with a filter in the usual \((a_l, b_m)\) form.

\[
y_n = x_n - \frac{3}{2}x_{n-1} + \frac{1}{2}x_{n-2} - y_{n-1} - \frac{1}{2}y_{n-2}
\]

First, we create the symmetric \((\alpha_l, \beta_m)\) form by moving all the \(y\) terms to the left side.

\[
y_n + y_{n-1} + \frac{1}{2}y_{n-2} = x_n - \frac{3}{2}x_{n-1} + \frac{1}{2}x_{n-2}
\]

Next, we write this as an equation for signals (rather than an equation for values in the time domain).

\[
\left(1 + \hat{z}^{-1} + \frac{1}{2} \hat{z}^{-2}\right)y = \left(1 - \frac{3}{2} \hat{z}^{-1} + \frac{1}{2} \hat{z}^{-2}\right)x
\]

Now we take the \(z\) transform of both sides, using the fundamental theorem \(zT(\hat{z}^{-1}x) = z^{-1}zT(x)\).

\[
\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)Y(z) = \left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right)X(z)
\]

This means that

\[
Y(z) = \frac{\left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right)}{\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)}X(z)
\]

But \(Y(z) = H(z)X(z)\) so we have found the transfer function of this filter:

\[
H(z) = \frac{\left(1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}\right)}{\left(1 + z^{-1} + \frac{1}{2} z^{-2}\right)}
\]

Multiplying top and bottom by \(z^2\) we obtain

\[
H(z) = \frac{\left(z^2 - \frac{3}{2} z + \frac{1}{2}\right)}{\left(z^2 + z + \frac{1}{2}\right)}
\]
which can be factored as follows:

\[ H(z) = \frac{(z - 1)(z - \frac{1}{2})}{(z + \frac{1}{2}(1 + i))(z + \frac{1}{2}(1 - i))} \]

which is a rational function (the ratio of two polynomials in \( z \)).

The zeros of the transfer function are the roots of the polynomial in the numerator. These are easily seen to be 1 and \( \frac{1}{2} \).

The poles of the transfer function are the roots of the polynomial in the denominator. A little algebra shows that these are \( -\frac{1}{2}(1 \pm i) \).

We see that there are zeros to the left of the y axis (low frequencies), including on at DC, and there are poles to the right of the y axis (high frequencies), so we can conclude that this is a high-pass filter. To understand this, you can input DC (\( x_n = \ldots+1+1+1+1 \ldots \)) and Nyquist (\( x_n = \ldots-1+1-1+1\ldots \)) to the original equation in the time domain and see what you get. Alternatively, look at the transfer function only on the unit circle by substituting \( z = e^{i\omega n} \) and find the frequency response \( H(\omega) \).

Finally, we can draw the pole-zero diagram of the filter, which determines the filter to within a gain factor.