Second set of exercises (systems)

1. Which of the following systems are filters? Explain.
   (a) $y_n = gx_n$ **of course**
   (b) $y_n = -x_n$ **of course**
   (c) $y_n = x_{n-1} + k$ **not linear**
   (d) $y_n = x_n + x_{-n}$ **not time-invariant - what happens if move n=0**
   (e) $y_n = x_{n+1}x_{n-1}$ **not linear**
   (f) $y_n = x_{ln}$ **not time-invariant**
   (g) $y_n = 0$ **of course**
   (h) $y_n = y_{n-1}$ **not linear**

2. Which of the following filters are causal? MA/AR/ARMA? LP/HP/BP/BS? (For the frequency response it is enough to check at DC and Nyquist.)
   (a) $y_n = -x_n$ **causal, MA, all-pass (gain=1 for all frequencies, phase shift=180 degrees)**
   (b) $y_n = x_n + x_{n-1}$ **causal, MA, low-pass (H(DC)=2, H(Nyquist)=0)**
   (c) $y_n = x_n - x_{n-1}$ **(first finite difference!) causal, MA, high-pass (H(DC)=0, H(Nyquist)=2)**
   (d) $y_n = x_n + x_{n+1}$ **noncausal, MA, low-pass (same as $\mathbf{x_n + x_{n-1}}$ except phase)**
   (e) $y_n = x_n - 0.99y_{n-1}$ **causal, AR, high-pass (zero blocker!) (H(DC)$\approx \frac{1}{2}$, H(Nyquist)=$\frac{1}{0.01}$)**
   (f) $y_n = x_n + x_{n-1} + \frac{1}{2}y_{n-1}$ **causal, ARMA, low-pass (H(DC)=4, H(Nyquist)=$\frac{4}{3}$)**

3. Show that an AR filter can be *unstable*, i.e., increase without limit even with a bounded input. Why can this happen for an AR filter but not an MA one? $\mathbf{y_n = x_n + 2y_{n-1}}$ **has IR $\mathbf{h_n = 1, 2, 4, 8, 16, \ldots}$. This can never happen when the output is a linear combination of inputs, but can happen when some of the output is fed back into the input. To understand why, consider a zero-delay feedback amplifier $\mathbf{y(t) = g_1x(t) + g_2y(t)}$, which means $\mathbf{y(t) = \frac{g_1}{1-g_2}x(t)}$ and the effective gain is infinite when $\mathbf{g_2 = 1}$. If the feedback has delay, e.g., $\mathbf{y(t) = g_1x(t) + g_2y(t-\tau)}$, then the explosion is not at DC, but at some frequency (the frequency of the whistle you hear when a microphone is too close to a speaker!).** What about an ARMA filter? **Try $\mathbf{y_n = x_n + x_{n-1} + 2y_{n-1}}$**

4. Show explicitly in the time domain that the two MA filters $y_n = a_{10}x_n + a_{11}x_{n-1} + a_{12}x_{n-2}$ and $y_n = a_{20}x_n + a_{21}x_{n-1} + a_{22}x_{n-2}$ commute. **straightforward substitution!**Show similarly that any two MA filters commute. $\mathbf{\sum_i a_i \sum_j b_j x_{n-i-j} = \sum_j b_j \sum_i b_i x_{n-j-i}}$ Show that the systems $y_n = x_n + a$ (which adds a DC term) and $y_n = x_n^2$ (which squares its input) do not commute. **since $\mathbf{(x_n + a)^2 \neq x_n^2 + a}$** Why is this possible from a DSP point of view? **not filters!** Find two systems that do not commute because only one of them is not linear. **$\mathbf{x_n^2}$ and gain** Find two systems that

do not commute because both are not time invariant. $\mathbf{nx_n}$ **and** $\mathbf{\Theta(n)x_n}$ **where** $\mathbf{\Theta(n)}$ **is the Heaviside step function**

5. Show that the analog echo system $y(t) = x(t) + hx(t - \tau)$ has no zeros in its frequency response for $|h| < 1$; i.e., there are no sinusoids that are exactly canceled out. **An attenuated sinusoid can never completely cancel out a nonattenuated one.** Find signals that *are* canceled out by this system. **We need the echo to be of opposite sign, so we use a sinusoid with frequency** ω **such that** $\omega\tau = \pi$**. Since** $|h| < 1$ **the echo is attenuated as compared to the original, so we need to compensate by making the signal's amplitude grow over time, so we use** $\mathbf{e^{\lambda t}}\sin(\omega\mathbf{t})$ **where we pick** $\lambda = -\frac{\ln(\mathbf{h})}{\tau}$ **(note that** $\ln(\mathbf{h}) < \mathbf{0}$ **so** $\lambda > \mathbf{0}$**).**

6. In a previous exercise you showed that a real sinusoid obeys a second order difference equation $s_n = \alpha s_{n-1} - s_{n-2}$ where $\alpha = 2\cos(\omega n)$. Based on this, show that $y_n = x_n + \alpha y_{n-1} - y_{n-2}$ can produce a sinusoidal output even with no input. **straightforward!**Yet this is a filter, and thus should not be able to create frequencies that are not in the input! Explain what is happening. **There is a pole. Were the input to be truly zero then there would be no output. But even a tiny bit of input energy at the pole's frequency is multiplied by an infinite gain, resulting in an non-zero output.**