A.11.1 Prove

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega t)\, d\omega$$

First, we should check whether the result is reasonable. If we integrate a sinusoid over any whole number of cycles we obviously get zero from symmetry. However, when $t = 0$ the cosine reduces to a constant DC and the integral diverges.

Now to the proof. Using equation (A.8)

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega t)\, d\omega \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{1}{2}\left(e^{\mathrm{i}\omega t} + e^{-\mathrm{i}\omega t}\right)\, d\omega \\
= & \frac{1}{4\pi}\left(\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\, d\omega + \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}\, d\omega\right) \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\, d\omega \\
= & \delta(t)
\end{aligned}$$

where we have noted that the second integral equals the first and used equation (A.73).

Were we to have asked the same question regarding sine rather than cosine we would have found something else.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \sin(\omega t)\, d\omega = 0$$

This is evident from a similar argument or by noting that sine is an odd function.

The question arises as to how the results could be so different. Since sine and cosine are the same to within a shift of $\frac{\pi}{2}$ we would expect infinite integrals to give identical results. In fact they do agree almost everywhere, since the δ function is zero almost everywhere. However, the argument breaks down when $t = 0$. The sine gives $\sin(0) = 0$ while the cosine gives $\cos(0) = 1$ which are *not* the same to within a shift. So both integrals are zero everywhere, except for at $t = 0$ where the cosine integral is infinite.